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C. 1

Abstract

A nonlinear boundary value problem is considered for the axisymmetric buckling of thin spherical shells subjected to uniform external pressure. The uniformly compressed spherical state is a solution of this problem for all values of the pressure. We prove, using Poincaré's method, that for pressures sufficiently near each simple eigenvalue of the linearized shell buckling theory, there is another (buckled) solution of the nonlinear problem. A convergent perturbation expansion is used to analyze the buckled solutions near the eigenvalues. For a limited range of caps, we also prove that one or three buckled solutions bifurcate from the multiple (double) eigenvalues depending on their order. The existence of a "lowest" intermediate buckling is established and precise upper and lower bounds are given on its magnitude.

1. Introduction

The surface of a thin elastic spherical cap is subjected to a uniform pressure, p, which is directed towards the cap's center of curvature. It is a well known experimental result, see e.g. [1], that as the pressure increases from zero the cap deforms only slightly from the spherical shape until a critical pressure $p = p_A$ is reached. Then the cap suddenly jumps, with relatively large deflections, into a non-spherical shape which we call the buckled state. The fundamental problem of shell buckling is to determine the mechanism which initiates the jumping and to obtain estimates of p_A.

Previous investigators[†] have assumed, as an approximation to experimental conditions, that the cap's edge is rigidly clamped, i.e. the displacement and change of slope are zero. In this paper edge conditions are considered for which the spherical shape (the unbuckled solution) is a possible solution of the nonlinear problem for all pressures. We refer to these as bifurcation buckling problems since other (buckled) solutions of the nonlinear problem may branch from the unbuckled solution.[*] For example, a bifurcation buckling problem, which we call Problem B, is obtained if the following conditions are specified on the edge of the cap: no rotation (clamped); zero transverse shear force, i.e. the edge is free to move normal to the spherical surface; and the meridional membrane stress is prescribed so that it is in equilibrium with the applied surface pressure. Other bifurcation problems are obtained, for example, by replacing the condition on the meridional stress in Problem B with a corresponding one on the meridional membrane displacement, or by permitting the cap to freely rotate instead of clamping it. The bifurcation problems are precisely formulated in Section 2. In Sections 3, 4 and 5 only Problem B is considered. However analogous results can be established for other bifurcation problems. Some of these are contained in the final section of the paper. We consider

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[†] See e.g. [1-7] and references contained therein.

[*] For the rigidly clamped cap, which we refer to as relaxation buckling, the spherical shape is a solution if and only if $p = 0$.

only axisymmetric deformations of the cap.

Bifurcation buckling problems for spherical caps were first considered in an approximate form in [8]. An equivalent of Problem B was treated in [5] where the linearized buckling theory was partially analyzed and approximate solutions of the nonlinear problem were obtained. The linear buckling theory had been previously discussed.[*]

The precise knowledge of the unbuckled state for the bifurcation problems, permits us to rigorously establish certain properties of the solution. In Section 3 we prove that for all P sufficiently near each simple eigenvalue of the linearized shell buckling theory[†], a solution of the nonlinear problem exists. Only one solution of the nonlinear problem branches from each simple eigenvalue. A related result is established in Section 4 for the other eigenvalues, which are all double, and for a limited range of caps. However, we find the surprising result that one or three solutions of the nonlinear problem bifurcate from the double eigenvalues depending on their order. The conjectured load-deformation curves for simple eigenvalues are sketched in Fig. 1. A perturbation expansion, which is valid near each eigenvalue, is used to prove that the curves in Fig. 1 have the form of the solid portion.

Our method of analysis is the Poincaré bifurcation theory used to prove the existence of periodic solutions of

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[*] Privately communicated to the author by W. Squires, 1958.

[†] Here P is a dimensionless parameter proportional to p, see Eq. (2.2a) below.

initial value problems. Previous applications of this technique to boundary value problems are given in [9 , 10] where related buckling problems for columns and circular plates are studied. In Section 3, where the simple eigenvalues are studied, the procedures employed are closely related to those given in [10]. Modifications are made in Section 4 to investigate the double eigenvalue case.

Friedrichs [11] proposed an energy mechanism and introduced the concept of an intermediate buckling load to explain the experimentally observed jumping of complete spheres from an unbuckled to a buckled state. Modifications and extensions of these ideas were subsequently proposed in [12]. As applied to the bifurcation buckling of caps, see e.g. Fig. 1, several intermediate buckling loads $P_M^{(n)}$ corresponding to different branches of the solution may exist. For a given branch, say bifurcating from the eigenvalue P_n, $P_M^{(n)}$ is defined as a load in the interval $P_L^{(n)} < P_M^{(n)} < P_n$ such that for all P in $P_L^{(n)} \leq P < P_M^{(n)}$ the unbuckled state has less potential energy than the corresponding buckled state and conversely for P in $P_M^{(n)} < P < P_n$. In Section 5 we establish, assuming that the potential energy has a minimum for every finite P and $\rho > 0$, the existence of a lowest intermediate buckling load P_M, see Fig. 1. It is shown to be bounded from below by the lowest buckling load of an "equivalent" flat circular plate buckling problem and bounded from above by the lowest eigenvalue, $\underline{P}$, of the linearized shell buckling theory. More accurate upper bounds, which are considerably less than $\underline{P}$, are rigorously

obtained in Section 5 by a minimization procedure. These bounds bracket P_M for a limited range of caps. It seems likely that with suitable modifications corresponding estimates of intermediate buckling loads can be obtained for the unsymmetric buckling of spherical caps and other shell bifurcation buckling problems. Some of the results of Section 5 are closely related to ones previously announced by Vorovich [13].

2. Formulation of the Boundary Value Problem

The elastic spherical cap is of thickness 2h and radius R and has a small angle of opening 2Λ; see Fig. 2 for the shell geometry. We consider the axisymmetric deformations of the cap that result from a uniform and inwardly directed pressure p. The non-vanishing mid-surface displacements, u and w which are in the meridional and normal directions to the shells mid-surface are therefore functions only of the polar angle θ. Both w and p are counted positive when directed towards the center of curvature. Nonlinear differential equations which describe the small finite axisymmetric deformations of thin spherical caps have been derived by several authors, e.g. [1,2,14-16]. These equations may be written as,

$$(2.1a) \qquad Gf(x) + \lambda f(x) = \rho g(x)[f(x) + 1] ,$$

$$(2.1b) \qquad Gg(x) = -\rho[f^2(x) + 2f(x)] .$$

Here we have used the dimensionless variables:

$$x \equiv \theta/\Lambda \ , \quad f(x) \equiv \frac{1}{R\theta}\frac{dw(\theta)}{d\theta} \ , \quad \rho = (\Lambda^2/C)(R/h) \ ,$$

(2.2a)

$$P \equiv (R/h)^2(p/2EC) \ , \quad \lambda \equiv P\rho \ , \quad C^2 \equiv 2/3(1-\nu^2) \ ,$$

and the linear differential operator G which is defined by

$$G\phi(x) \equiv x^{-3}[x^3\phi'(x)]' \ ,$$

where a prime indicates differentiation with respect to x. Here E is Young's modulus and ν is Poisson's ratio.*

The "excess" stress function g(x) in (2.1) is defined such that the meridional and circumferential membrane stresses σ_θ and σ_ϕ and the corresponding dimensionless stresses Σ_θ and Σ_ϕ are given by

$$\Sigma_\theta(x) \equiv (R/h)(2/EC)\sigma_\theta(\theta) = g(x) - P \ ,$$

(2.2b)

$$\Sigma_\phi(x) \equiv (R/h)(2/EC)\sigma_\phi(\theta) = [xg(x) - xP]' \ .$$

The outer surface bending stresses σ_θ^o and σ_ϕ^o, and the corresponding dimensionless stresses Σ_θ^o and Σ_ϕ^o are given in terms of f(x) by

$$\Sigma_\theta^o(x) \equiv (R/h)(2/3EC^2)\sigma_\theta^o(\theta) = xf'(x) + (1+\nu)f(x) \ ,$$

(2.2c)

$$\Sigma_\phi^o(x) \equiv (R/h)(2/3EC^2)\sigma_\phi^o(\theta) = \nu xf'(x) + (1+\nu)f(x) \ .$$

The independent variable f(x) is related to the slope of the deformed middle surface of the cap with respect to the initial spherical shape. Thus if $f(x) \equiv 0$ for a given deformation the deformed middle surface is also spherical. We

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[*] The independent variables $\alpha(x)$ and $\gamma(x)$ employed in previous papers [2,3] are related to the present variables by,

$$\alpha(x) = x[f(x) + 1] \ , \quad \gamma(x) = x[g(x) - P] \ .$$

refer to ρ, which is defined in (2.2a), as the geometrical parameter and P and λ as either pressure or loading parameters.

To complete the formulation conditions at the center $x = 0$ and the edge $x = 1$ are required. From the symmetry of the deformation and the regularity of the membrane and bending stresses at the origin we obtain with aid of (2.1) that,

$$f'(0) = g'(0) = 0 \ . \tag{2.3}$$

The edge of the cap is restrained from rotating so that,

$$f(1) = 0 \ . \tag{2.4a}$$

In addition, we assume that at $x = 1$ the transverse shear force vanishes and $\Sigma_\theta(1)$ is specified so that the cap is in equilibrium with the applied pressure. This yields the boundary condition

$$g(1) = 0 \ . \tag{2.4b}$$

The bifurcation buckling problem, Problem B, is defined as the boundary value problem consisting of the differential equations (2.1) and the boundary conditions (2.3) and (2.4).

Other boundary conditions can be specified at $x = 1$ to yield bifurcation buckling problems for clamped caps, e.g.

$$f(1) = 0 \ , \quad g'(1) + (1 - \nu)g(1) = 0 \ . \tag{2.5}$$

These conditions imply that on the edge the transverse shear force vanishes and the meridional displacement u (or equivalently the horizontal displacement) is proportional to P The proportionality constant is determined so that the spherical form is a possible solution for all P.

3. The Existence of Buckled Solutions.

A solution of Problem B that is valid for all finite P and ρ is the unbuckled solution, $f(x) \equiv g(x) \equiv 0$. This corresponds to a state of uniform compression in which

$$\Sigma_\theta (x) = \Sigma_\phi (x) = -P ,$$

and the deformed middle surface remains spherical.

The existence of buckled states will now be established using Poincaré's method. Specifically, we prove in this section that for every $\rho > 0$ and for every positive integer n there exists a buckled solution when P is in a sufficiently small and full interval about the n-th simple eigenvalue, P_n, of the linearized shell buckling theory. This is in contrast to the buckled circular plate [10] where the load must be slightly greater than each eigenvalue of the corresponding linear buckling theory. We also show that for P sufficiently near P_n the solution has n-1 internal nodes. In the following section the solutions of Problem B near the other eigenvalues of the linearized theory, which are all double, are studied.

To employ Poincaré's method parameters ε and δ and new independent variables $y(x)$ and $z(x)$ are defined by,

$$(3.1a) \qquad \varepsilon \equiv \lim_{x \to 0} [xf(x)]' \quad , \qquad \delta \equiv \varepsilon^{-1} \lim_{x \to 0} [xg(x)]' \ ,$$

$$(3.1b) \qquad y(x) \equiv \varepsilon^{-1} xf(x) \quad , \qquad z(x) \equiv \varepsilon^{-1} xg(x) \ .$$

The differential equations (2.1) of Problem B are then given by,

$$(3.2a) \qquad Hy(x) + P\rho y(x) = \rho z(x)[1 + \varepsilon y(x)/x] \ ,$$

$$(3.2b) \qquad Hz(x) = -\rho[\varepsilon y^2(x)/x + 2y(x)] \ ,$$

where H is the linear differential operator,

$$(3.2c) \qquad H\phi(x) = \{x^{-1}[x\phi(x)]'\}' \ .$$

The initial value problem, $\mathcal{J}$, is defined by the differential equations (3.2) and the initial conditions:

$$(3.3a) \qquad y(0) = 0 \quad , \quad y'(0) = 1 \ ,$$

$$(3.3b) \qquad z(0) = 0 \quad , \quad z'(0) = \delta .$$

For all finite P, ε, δ and $\rho > 0$, Problem $\mathcal{J}$ has a unique solution in the interval $0 \leq x \leq 1$ which is analytic in

P, ε and δ. This result can be proved by the same methods used in [10] to establish an analogous result for the buckling of circular plates and hence the proof is not given here. For fixed $\rho > 0$ values of the parameters P, ε and δ are sought such that the solution, $[y(x;P,\varepsilon,\delta),\ z(x;P,\varepsilon,\delta)]$ of $\mathcal{A}$ satisfies the boundary conditions

$$(3.4)\qquad y(1;P,\varepsilon,\delta) = z(1;P,\varepsilon,\delta) = 0 .$$

If such a choice of parameters is possible it then follows from (3.1b) that $f(x) = \dfrac{\varepsilon y(x;P,\varepsilon,\delta)}{x}$ and $g(x) = \dfrac{\varepsilon z(x;P,\varepsilon,\delta)}{x}$ is a solution of B.

A solution of $\mathcal{A}$ which satisfies the boundary conditions (3.4) is obtained by choosing the special parameter value, $\varepsilon = 0$ in (3.2). The resulting initial value problem for the functions $y = y(x;P,0,\delta)$, $z = z(x;P,0,\delta)$ has solutions which satisfy (3.4) if and only if

$$\left.\begin{aligned}
&(3.5a)\quad P(\rho) = P_n(\rho) \equiv 2\rho/\lambda_n^2 + \lambda_n^2/\rho \geq 2\sqrt{2}\\
&(3.5b)\quad \delta(\rho) = \delta_n(\rho) \equiv 2\rho/\lambda_n^2
\end{aligned}\right\}\quad n = 1,2,\ldots,$$

where λ_n is the n-th zero of the Bessel function $J_1(x)$. The quantities $P_n(\rho)$ are the eigenvalues (or buckling loads) of the linearized buckling theory and they are simple eigenvalues if $\rho \neq \rho_{mn}$ and $P \neq P_n(\rho_{mn}) = P_m(\rho_{mn})$ where,

$$(3.6)\qquad \rho_{mn} \equiv 2^{-1/2}\omega_m\omega_n\ , \quad m,n = 1,2,\ldots, \quad m \neq n\ .$$

The solutions are the eigenfunctions (or buckling modes) of the linearized theory and for simple eigenvalues are given by[*]

$$(3.7)\qquad \left.\begin{aligned} y(x;P_n,0,\delta_n) &\equiv y^{(n)}(x) = (A_n/\delta_n)J_1(\omega_n x)\ , \\ z(x;P_n,0,\delta_n) &\equiv z^{(n)}(x) = A_nJ_1(\omega_n x)\ , \end{aligned}\right\} \quad n = 1,2,\ldots,$$

where

$$A_n \equiv 2\rho/\omega_n^3 J_1'(0)\ .$$

The corresponding solutions of Problem B reduce to the unbuckled solution.

Thus for each finite $\rho > 0$ there are a denumerably infinite number of roots $[P_n,0,\delta_n]$, $n = 1,2,\ldots$ of (3.4). According to the implicit function there are other roots of (3.4) near $[P_n,0,\delta_n]$ if the appropriate Jacobian

$$(3.8)\qquad J \equiv z_\delta^{(n)}(1)y_P^{(n)}(1) - z_P^{(n)}(1)y_\delta^{(n)}(1) \neq 0\ , \qquad n = 1,2,\ldots\ .$$

Here we have used the notation

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[*] For simplicity, the explicit dependence of the solutions on ρ is suppressed here and in the remainder of the section.

$$(3.9) \qquad \frac{\partial}{\partial \delta}\, y(x;P,\varepsilon,\delta)\Bigg|_{\substack{P=P_n\\ \varepsilon=0\\ \delta=\delta_n}} \equiv y_\delta^{(n)}(x)\ , \text{ etc.} \qquad n = 1,2,\ldots\ .$$

Since y and z are analytic in P and δ, the initial value problems (variational problems) satisfied by y_P, y_δ, z_P and z_δ can be obtained by formal differentiation of $\mathcal{A}$ with respect to P and δ. Thus we obtain from (3.2) and (3.3) using (3.7) and (3.9),

$$\left.\begin{array}{lll}
(3.10a) & Hy_P^{(n)}+P_n\rho y_P^{(n)} = \rho[z_P^{(n)}-y^{(n)}]\ , & y_P^{(n)}(0) = y_P^{(n)\prime}(0)=0, \\
(3.10b) & Hz_P^{(n)} = -2\rho y_P^{(n)}\ , & z_P^{(n)}(0) = z_P^{(n)\prime}(0)=0; \\
(3.11a) & Hy_\delta^{(n)} + P_n\rho y_\delta^{(n)} = \rho z_\delta^{(n)}\ , & y_\delta^{(n)}(0) = y_\delta^{(n)\prime}(0)=0, \\
(3.11b) & Hz_\delta^{(n)} = -2\rho y_\delta^{(n)}\ , & z_\delta^{(n)}(0) = 0, z_\delta^{(n)\prime}(0)=1,
\end{array}\right\} n=1,2,\ldots$$

The explicit solution of (3.11) (not presented) yields, at x = 1, the relations

$$(3.12) \qquad z_\delta^{(n)}(1) = 2\delta_n^{-1} y_\delta^{(n)}(1) \begin{cases} \neq 0 \text{ if } \rho \neq \rho_{mn} \\ = 0 \text{ if } \rho = \rho_{mn} \end{cases}, \quad m,n=1,2,\ldots,\ m\neq n.$$

Therefore it follows from (3.8) and (3.12) that

$$(3.13) \qquad J = y_{\delta}^{(n)}(1)\left[2\delta_n^{-1} y_P^{(n)}(1) - z_P^{(n)}(1)\right] .$$

Hence, $J = 0$ if $\rho = \rho_{mn}$ and $P = P_n$ and nothing can be concluded concerning the existence of neighboring roots to $[P_n(\rho_{mn}), 0, \delta_n(\rho_{mn})]$. This case is considered in the following section. To prove that $J \neq 0$ if $\rho \neq \rho_{mn}$, $P \neq P_n$ we assume the contrary, i.e. let,

$$(3.14) \qquad y_P^{(n)}(1) = 2^{-1}\delta_n z_P^{(n)}(1) = \alpha$$

where α is an arbitrary real constant. Then, $[y_P^{(n)}(x), z_P^{(n)}(x)]$ is a solution of (3.10) that satisfies the boundary conditions (3.14). A solution of this boundary value problem exists if and only if the inhomogeneous terms are orthogonal to every solution of the homogeneous adjoint problem. This leads to the condition that for all α

$$\int_0^1 J_1^2(\omega_n x) x dx = 0 ,$$

which is impossible and hence $J \neq 0$.

Hence, by the implicit function theorem and the analyticity of y and z in the parameters, (3.4) can be uniquely solved for P and δ as analytic functions of ε in some sufficiently small neighborhood of each root $[P_n, 0, \delta_n]$, $\rho \neq \rho_{mn}$. The solutions are the analytic functions

$$(3.15)\quad P = P_{(n)}(\varepsilon),\quad \delta = \delta_{(n)}(\varepsilon),\quad |\varepsilon| < \varepsilon_n,\quad n = 1,2,\dots ,$$

which satisfy the conditions

$$P_{(n)}(0) = P_n\ ,\quad \delta_{(n)}(0) = \delta_n\ .$$

For sufficiently small ε

$$(3.16)\quad \begin{aligned} f(x;\varepsilon) &= \varepsilon x^{-1} y_n(x;\varepsilon) \equiv \varepsilon x^{-1} y(x;P_{(n)}(\varepsilon),\varepsilon,\delta_{(n)}(\varepsilon))\ , \\ g(x;\varepsilon) &= \varepsilon x^{-1} z_n(x;\varepsilon) \equiv \varepsilon x^{-1} z(x;P_{(n)}(\varepsilon),\varepsilon,\delta_{(n)}(\varepsilon)) \end{aligned}$$

are solutions of B.

The solutions of B near $\varepsilon = 0$ are now considered. Since $P_{(n)}(\varepsilon)$, $\delta_{(n)}(\varepsilon)$, $y_n(x;\varepsilon)$ and $z_n(x;\varepsilon)$ are analytic functions there, they have convergent expansions in some interval $|\varepsilon| < \varepsilon_n^o$ given by:

$$(3.17)\quad \begin{aligned} P_{(n)}(\varepsilon) &= P_n + \sum_{i=1}^{\infty} P_i^{(n)} \varepsilon^i\ , & \delta_{(n)}(\varepsilon) &= \delta_n + \sum_{i=1}^{\infty} \delta_i^{(n)} \varepsilon^i \\ y_n(x;\varepsilon) &= y^{(n)}(x) + \sum_{i=1}^{\infty} y_i^{(n)}(x)\varepsilon^i, & z_n(x;\varepsilon) &= z^{(n)}(x) + \sum_{i=1}^{\infty} z_i^{(n)}(x)\varepsilon^i . \end{aligned}$$

The expansion coefficients are determined in the usual way by substituting (3.17) into (3.2) and (3.4) and equating the coefficients of each power of ε. Each of the resulting system of linear boundary value problems for the coefficients $[y_i^{(n)}, z_i^{(n)}]$ has a solution if and only if the appropriate

orthogonality condition is satisfied. For the first two of these problems with $i = 1,2$ the orthogonality conditions yield

$$(3.18a)\quad P_1^{(n)} = \frac{6\rho}{\omega_n^3 J_1'(0) J_2^2(\omega_n)} \int_0^1 J_1^3(\omega_n x)\,dx > 0 ,$$

$$(3.18b)\quad P_2^{(n)} = \frac{4\rho}{(\omega_n J_2(\omega_n))^2} \int_0^1 [2y_1^{(n)}(x) + \delta_n^{-1} z_1^{(n)}(x)]\, J_1^2(\omega_n x)\,dx$$

$$- \frac{2J_1'(0)\omega_n P_1^{(n)}}{J_2^2(\omega_n)} \int_0^1 y_1^{(n)}(x) J_1(\omega_n x)\,x\,dx .$$

Thus it follows from the first of (3.17) and (3.18a) that for each n solutions of B must exist for all P in some small interval about P_n.

For sufficiently small ε we have from (3.17)

$$y_n(x;\varepsilon) = y^{(n)}(x) + O(\varepsilon) .$$

Hence in the interval $0 < x < 1$, y_n has the same number of zeros as $y^{(n)}$, or y_n has $n-1$ simple zeros.

Approximations of the lower buckling load, $P_{(n)L}$, for each branch may be obtained by truncating the series in the first of (3.17) and from the condition that $\frac{dP_{(n)}(\varepsilon)}{d\varepsilon} = 0$ at the lower buckling load. These approximations may be valid only if the expansions (3.17) converge for sufficiently large ε. For example if 2 terms are retained in the series so that

$$P_n(\varepsilon) \approx P_n + P_1^{(n)}\varepsilon + P_2^{(n)}\varepsilon^2$$

then

$$P_{(n)L} \approx P_n - \frac{P_1^{(n)2}}{4P_2^{(n)}} .$$

4. The Existence of Buckled Solutions Near Double Eigenvalues

We now consider $\rho = \rho_{mn}$ and the double eigenvalues,

$$(4.1)\quad P = P_m(\rho_{mn}) = P_n(\rho_{mn}) \equiv \omega_m^2 + \omega_n^2 , \quad m,n = 1,2,\ldots, m \neq n .$$

The solutions of the initial value problem (3.2), (3.3) with $\varepsilon = 0$, $\rho = \rho_{mn}$ and $P = P_m(\rho_{mn})$, that satisfy the boundary conditions (3.4) are the eigenfunctions*

$$y(x;P_m(\rho_{mn}),0,\delta)=y^{(mn)}(x;\delta)=(a_{mn}/\delta_{mn})J_1(\omega_m x)+(a_{nm}/\delta_{nm})J_1(\omega_n x),$$

(4.2a)

$$z(x;P_m(\rho_{mn}),0,\delta) = z^{(mn)}(x;\delta) \equiv a_{mn}J_1(\omega_m x) + a_{nm}J_1(\omega_n x) ,$$

where δ is an arbitrary real number and

$$\delta_{mn} \equiv 2^{1/2}\omega_n/\omega_m ,$$

(4.2b)

$$a_{mn}(\delta) \equiv \frac{2^{1/2}\omega_n}{J_1'(0)(\omega_n^2 - \omega_m^2)} [(\delta/\delta_{nm}) - 1] .$$

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* Here, and in all subsequent equations in this section, the indicies $m,n = 1,2,\ldots$, $m \neq n$, and we shall hereafter omit explicit reference to this.

The double eigenvalue case may be of especial physical significance. Consider the variation with ρ of $\underline{P}(\rho)$ which is defined by

$$(4.3) \qquad \underline{P}(\rho) \equiv \min_{n} P_n(\rho) .$$

There are local minimums of $\underline{P}(\rho)$ at $P = 2^{3/2}$, $\rho = 2^{1/2}\omega_n^2$, $n = 1,2,\ldots$. At $\rho = \rho_{n,n+1}$ $\frac{d\underline{P}}{d\rho}$ is discontinuous or there are "peaks" in the curve of $\underline{P}(\rho)$ at the double eigenvalues. A corresponding peaking behavior was previously noted [2,17] in the experimental buckling loads obtained by Kaplan and Fung*[1]. Although the experimental boundary conditions probably differ from those of Problem B, this phenomenon may be related to the occurrence of a double eigenvalue.

Since the Jacobian in (3.8) vanishes if $\rho = \rho_{mn}$ and $P = P_n$, the procedure given in Section 3 must be modified to investigate this case. The modifications are essentially extensions of the methods used in the Poincaré theory when the variational problem possesses periodic solutions, see [18]. A new parameter κ is introduced and modified initial value problems, $\mathcal{S}_{mn}$, for the functions $y_{mn}(x;P,\varepsilon,\delta,\kappa)$ and $z_{mn}(x;P,\varepsilon,\delta,\kappa)$ are defined by

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* For example, the experimental peaks occur at $\rho \approx 20$, 56 and $\rho_{12} \approx 19$, $\rho_{23} \approx 50.5$.

$$Hy_{mn} + P\rho_{mn}y_{mn} - \rho_{mn}z_{mn} = \varepsilon\rho_{mn}x^{-1}y_{mn}z_{mn} + \kappa y^{(mn)} ,$$

(4.4) $$Hz_{mn} + 2\rho_{mn}y_{mn} = -\varepsilon\rho_{mn}x^{-1}y_{mn}^{2} + \kappa z^{(mn)} ,$$

$$y_{mn} = 0, \quad y'_{mn} = 1, \quad z_{mn} = 0, \quad z'_{mn} = \delta, \text{ for } x = 0 .$$

We can show, as in the preceding section that the unique solution of $\mathcal{L}_{mn}$ is an analytic function of the parameters P, ε, δ and κ. The parameter values are to be determined such that the solutions of $\mathcal{L}_{mn}$ satisfy the boundary conditions,

(4.5) $$y_{mn}(1;P,\varepsilon,\delta,\kappa) = z_{mn}(1;P,\varepsilon,\delta,\kappa) = 0 .$$

For the special parameter values $\varepsilon = \kappa = 0$ and $P = P_m = P_n$ the solutions of $\mathcal{L}_{mn}$ are $y_{mn}(x;P_m(\rho_{mn}),0,\delta^{o}_{mn},0) = y^{(mn)}(x;\delta^{o}_{mn})$, $z_{mn}(x;P_m(\rho_{mn}),0,\delta^{o}_{mn},0) = z^{(mn)}(x;\delta^{o}_{mn})$, and they satisfy (4.5) for all bounded and real δ^{o}_{mn}. The implicit function theorem is used to determine other roots of (4.5) near each root $[P_m(\rho_{mn}),0,\delta^{o}_{mn},0]$. Proceeding as in Section 3 and using the appropriate variational problems we can show that the Jacobian of (4.5) with respect to P and κ and evaluated at each root does not vanish for all $\delta^{o}_{mn} \neq \delta_{mn}, \delta_{nm}$. Thus there are solutions of (4.5),

(4.6) $$P = P_{mn}(\varepsilon,\delta) , \quad \kappa = \kappa_{mn}(\varepsilon,\delta) ,$$

which are analytic functions of ε and δ in some sufficiently small neighborhood of each root $[P_m(\rho_{mn}),0,\delta^0_{mn},0]$, for which $\delta^0_{mn} \neq \delta_{mn}$, δ_{nm} and satisfy the conditions

$$(4.7)\qquad P_{mn}(0,\delta^0_{mn}) = P_m = P_n \ , \quad \kappa_{mn}(0,\delta^0_{mn}) = 0 \ .$$

Hence the modified problems (4.4), (4.5) have solutions $y_{(mn)}(x;\varepsilon,\delta) = y_{mn}(x;P_{mn}(\varepsilon,\delta),\varepsilon,\delta,\kappa_{mn}(\varepsilon,\delta))$, $z_{(mn)}(x;\varepsilon,\delta) \equiv z_{mn}(x;P_{mn}(\varepsilon,\delta),\varepsilon,\delta,\kappa_{mn}(\varepsilon,\delta))$ which are analytic in ε and δ for sufficiently small $|\varepsilon|$ and $|\delta-\delta^0_{mn}|$.

Solutions of the modified problems are solutions of the original problem (3.2), (3.3) and (3.4) if δ and ε are chosen so that the bifurcation equations [18]

$$(4.8)\qquad \kappa_{mn}(\varepsilon,\delta) = 0 \ ,$$

are satisfied. Thus if (4.8) have solutions $\delta = \delta_{mn}(\varepsilon)$ which satisfy the conditions

$$(4.9)\qquad \delta_{mn}(0) = \delta^0_{mn} \ ,$$

then

$$(4.10)\qquad \begin{aligned} f_{mn}(x;\varepsilon) &= \varepsilon x^{-1} y_{(mn)}(x;\varepsilon,\delta_{mn}(\varepsilon)) \ , \\ g_{mn}(x;\varepsilon) &= \varepsilon x^{-1} y_{(mn)}(x;\varepsilon,\delta_{mn}(\varepsilon)) \ , \end{aligned}$$

are solutions of Problem B with $\rho = P_{mn}$.

To investigate the solutions of (4.8) we employ the analyticity of y, z, P and κ and expand them in the forms:

$$y_{(mn)}(x;\varepsilon,\delta) = y^{(mn)}(x;\delta) + \sum_{k=1}^{\infty} y_k^{(mn)}(x;\delta)\,\varepsilon^k ,$$

$$z_{(mn)}(x;\varepsilon,\delta) = z^{(mn)}(x;\delta) + \sum_{k=1}^{\infty} z_k^{(mn)}(x;\delta)\,\varepsilon^k ,$$

(4.11)

$$P_{mn} = P_m + \sum_{k=1}^{\infty} P_k^{(mn)}(\delta)\,\varepsilon^k , \quad \kappa_{mn} = \varepsilon \sum_{k=1}^{\infty} \kappa_k^{(mn)}(\delta)\,\varepsilon^{k-1} ,$$

where we have used the last of (4.7). The series in (4.11) converge in some interval $|\varepsilon| < \varepsilon_{mn}^{o}$. The coefficients in the expansions are determined by substituting (4.11) into (4.4) and (4.5). The linear boundary value problems for $y_1^{(mn)}$ and $z_1^{(mn)}$, obtained from the coefficients of ε, possess solutions, if and only if the inhomogeneous terms satisfy the appropriate orthogonality condition. This condition determines $\kappa_1^{(mn)}$ and $P_1^{(mn)}$ as

$$\kappa_1^{(mn)}(\delta) = (\delta_{mn}^2 - \delta_{nm}^2)^{-1}(\delta_{mn}A_{mn} - \delta_{nm}A_{nm}) ,$$

(4.12)

$$P_1^{(mn)}(\delta) = (\omega_m^2 + \omega_n^2)^{-1}(A_{mn} + A_{nm})$$

where,

$$A_{mn} = \frac{2\delta_{mn}\omega_{mn}}{a_{mn}J_2^2(\omega_m)}\{3\delta_{mn}^{-2}\phi_{mm}a_{mn}^2 + 2^{-1}(4+\delta_{nm}^2)\phi_{mn}a_{mn}a_{nm} + (\delta_{nm}^{-2}+1)\phi_{nm}a_{nm}^2\} ,$$

(4.13)

$$\phi_{mn} = \int_0^1 J_1^2(\omega_m x)\,J_1(\omega_n x)\,dx .$$

Using the last of (4.11), the bifurcation equations (4.8) reduce to

(4.14) $$\sum_{k=1}^{\infty} \kappa_k^{(mn)}(\delta)\varepsilon^{k-1} = 0 \ .$$

If (4.14) have continuous solutions $\delta = \delta_{mn}(\varepsilon)$ with $\delta_{mn}(0) = \delta_{mn}^o$ then δ_{mn}^o must satisfy the bifurcation conditions

(4.15) $$\kappa_1^{(mn)}(\delta_{mn}^o) = 0 \ .$$

Conversely, if δ_{mn}^o satisfy the bifurcation conditions (4.15) and if

(4.16) $$\frac{d\kappa_1^{(mn)}(\delta_{mn}^o)}{d\delta} \neq 0$$

then the implicit function theorem is applicable to (4.14) and it implies that the bifurcation equations can be solved in some sufficiently small neighborhood of $\delta = \delta_{mn}^o$, $\varepsilon = 0$. The solutions $\delta = \delta_{mn}(\varepsilon)$ are analytic functions for sufficiently small ε and satisfy $\delta_{mn}(0) = \delta_{mn}^o$.

Inserting (4.12) and (4.13) into (4.15) and (4.16) and defining ξ_{mn} by

(4.17) $$\xi_{mn} = \frac{a_{mn}(\delta_{mn}^o)}{a_{nm}(\delta_{mn}^o)} \ ,$$

the bifurcation conditions reduce to the cubics:

$$(4.18) \qquad \xi_{mn}^3 + E_{mn}\xi_{mn}^2 + G_{mn}E_{nm}\xi_{mn} + G_{mn} = 0 \ ,$$

and the "solvability conditions" (4.16) reduce to:

$$(4.19) \quad \xi_{mn}^4 + 2\left(\frac{\omega_n}{\omega_m}\right)^3 \xi_{mn}^3 + \left[\left(\frac{\omega_n}{\omega_m}\right)^3 E_{mn} - G_{mn}E_{nm}\right]\xi_{mn}^2 - 2G_{mn}\xi_{mn} - \left(\frac{\omega_n}{\omega_m}\right)^3 G_{mn} \neq 0 \, .$$

Here we have used the notation,

$$E_{mn} \equiv \frac{(\omega_m/\omega_n)^4}{(1+\xi_{mn}^2)\phi_{mn}}\left[2^{-1}\xi_{nm}^2(4+\xi_{mn}^2)\phi_{nm} - 3\,\frac{J_2^2(\omega_n)}{J_2^2(\omega_m)}\,\phi_{mm}\right] ,$$

(4.20)

$$G_{mn} \equiv -(\omega_m/\omega_n)^8 \, \frac{(1+\xi_{nm}^2)}{(1+\xi_{mn}^2)} \, \frac{J_2^2(\omega_n)}{J_2^2(\omega_m)} \, \frac{\phi_{nm}}{\phi_{mn}} \, .$$

Thus we conclude from (4.18) that for each m and n there is at least one real ξ_{mn}^0 that satisfies the bifurcation condition and $\xi_{mn}^0 = \xi_{mn}$, ξ_{nm} are not roots of (4.18).

If the explicit values of the integrals ϕ_{mn}, defined in (4.13) are known then the roots of (4.18) can be determined using standard formulae. These integrals were numerically determined using Simpson's Rule with 500 mesh points* for m and n in the range $1 \leq m, n \leq 25$. The Bessel functions were evaluated using the procedure of [19] for arguments between zero and eight and appropriate asymptotic formulae were used

---

* Several of the integrals were evaluated using 1000 mesh points. No significant difference was observed with the results of the 500 point mesh. Double precision arithmetic was employed and all calculations were preformed on the IBM 7094 computer at the AEC Computing and Applied Mathematics Center of the Courant Institute of Mathematical Sciences. The author is indebted to Dr. F. Bauer for conducting the computation.

for larger arguments. We found that all the roots of (4.18), for m and n in the range[†] $1 \leq m, n \leq 25$, satisfied (4.19) well within the accuracy of the computation. In fact, the left side of (4.19) was usually quite large. Hence we have shown that there is at least one solution of Problem B for P in some sufficiently small neighborhood of each double eigenvalue with m and n in the above range. We conjecture that there are solutions near each double eigenvalue for all m,n=1,2,... . We also find that for the following indicies (4.18) has three real roots:

1) $n=m+1$, $3 \leq n \leq 25$, n odd,
2) $n=m+3$, $14 \leq n \leq 25$, n odd,
3) $n=2m+1$, $2 \leq n \leq 25$,
4) $n=2m+3$, $6 \leq n \leq 25$,
5) $n=2m+5$, $8 \leq n \leq 25$.

Thus in a sufficiently small neighborhood of the double eigenvalues with the above indicies there are three solutions of Problem B. Hence, as $\rho \longrightarrow \rho_{mn}$ the m-th and n-th branches coalesce and when $\rho = \rho_{mn}$ one solution may be "destroyed" or a third solution may be "created" depending on the values of the indicies m and n.

[†] By symmetry we need only consider $n > m$.

5. The Intermediate Buckling Load

It is convenient to reformulate Problem B to establish the existence of the intermediate buckling load P_M, or equivalently $\lambda_M = \rho P_M$, and to obtain upper and lower bounds on its magnitude. Equation (2.1b) is integrated and the second of (2.3) is used to obtain

$$(5.1) \qquad g'(x) = -\rho x^{-3} \int_0^x [f^2(x_0) + 2f(x_0)] x_0^3 dx_0 \ .$$

Then, using (2.4b), g(x) is given as a functional of f(x) by

$$(5.2) \qquad g(x) = \int_1^x g'(x_1) dx_1 \ .$$

The boundary value problem, B, is reformulated as Problem B' as follows: To find a function f(x) which possesses a continuous second derivative and satisfies the differential equation (2.1a) and the boundary conditions

$$(5.3) \qquad f'(0) = f(1) = 0 \ .$$

The function g(x) in (2.1a) is defined by (5.1) and (5.2). These equations imply that g(x) satisfies (2.1b) and $g'(0) = g(1) = 0$. This statement of the boundary value problem is similar to the one introduced in [20] in a study of the buckling of circular plates.

The "energy" functional V is defined as,

$$(5.4) \quad V[f(x);\lambda,\rho] \equiv \int_0^1 [2(f'^2(x) - \lambda f^2(x)) + g'^2(x)] x^3 dx \ ,$$

5. [illegible] $f(t)$.

[illegible]

(5.1) [illegible]

(5.2) [illegible]

[illegible]

[illegible]

(5.3) [illegible]

the function $g(t)$ [illegible] in (5.1) and (5.3), [illegible] only that $g(t)$ satisfies (5.10) and $g'(t) = f(t) = 0$. This [illegible] problem is similar to the one [illegible] in [10] in a study of the [illegible].

The "energy" functional E is defined as.

(5.6) $E(f(t))$ [illegible] $[f'(t)]^2 + [illegible] [f(t)]^2 \, dx$, where $g'(x)$ is considered as a functional of $f(x)$ defined by (5.1). Thus V is a functional of $f(x)$ only and it is proportional to the difference between the potential energies of a buckled and the unbuckled state.

The relationship between B' and V is obtained by first defining a class of functions. $f(x)$ is contained in A or is an A-function if in the interval $0 \leq x \leq 1$ it is continuous and satisfies (5.3), $f'(x)$ is an L_2 function and all integrals in (5.1), (5.2) and (5.4) exist. Then it is easy to show that if $f(x)$ is a solution of B' it makes V stationary with respect to all A-functions. The converse of this result can be proved using the methods outlined in [20]. The converse states that if $f(x)$ makes V stationary (or minimizes V) then $f''(x)$ is continuous and $f(x)$ solves B. Here $f_o(x) \in A$ is said to minimize V for a fixed λ and ρ if $V[f_o(x)] \leq V[f(x)]$ for all $f(x) \in A$.

The result concerning the existence of λ_M is now stated as the

THEOREM. *If for every finite* ρ *and* λ *there is an* A-*function which minimizes* V, *then* $\lambda_M(\rho)$ *exists and it is in the interval* $\omega_1^2 \leq \lambda_M(\rho) < \underline{\lambda} \equiv \rho\underline{P}(\rho)$, *where* $\underline{P}(\rho)$ *is defined in* (4.3), *and* ω_1 *is the first zero of* $J_1(x) = 0$.

The theorem is a direct consequence of the lemmas given below and the following inequality, which is easily demonstrated using classical methods [21],

$$(5.5)\quad \int_0^1 [f'^2(x) - \omega_1^2 f^2(x)]x^3dx \geq 0 \ , \quad \text{for all } f(x) \in A \ .$$

Here ω_1^2 is also the lowest eigenvalue of the linearized buckling problem for the symmetric deformations of radially compressed and clamped circular plates:

$$(5.6)\quad Gf(x) + \lambda f(x) = 0 \ , \quad f'(0) = f(1) = 0 \ .$$

We call (5.6) the "equivalent" circular plate problem. The inequality (5.5) and the functional V given in (5.4) and (5.1) immediately yield

LEMMA 1. *If* $\lambda \leq \omega_1^2$ *then only* $f(x) \equiv 0$ *minimizes* V.

The form of the functional V and the properties of the eigenvalues (3.5a) and eigenvectors (3.7) of the linear shell buckling problem yield

LEMMA 2. *If* $\lambda = \underline{\lambda}(\rho)$ *then there exist for each* $\rho > 0$ *a function* $f(x) \in A$ *such that* $V[f(x)] < 0$.

The proof of this lemma is obtained by a simple calculation of $V[f_n(x)]$ where $f_n(x) \equiv B_n J_1(\omega_n x)/x$ are eigenfunctions of the linear theory and B_n are constants restricted to lie in the ranges $0 > B_n > -2\beta_n/\alpha_n^2$, $n = 1,2,\ldots$. The constants α_n and β_n are defined by

$$\alpha_n^2 \equiv \rho^2 \int_0^1 \left[\int_0^x J_1^2(\omega_n x')x'dx' \right] x^{-3}dx \ ,$$

$$\beta_n \equiv 2\rho^2 \int_0^1 \left[\int_0^x J_1^2(\omega_n x')x'dx' \right]\left[\int_0^x J_1(\omega_n x')x'^2dx' \right] x^{-3}dx > 0 \ .$$

Lemma 1 implies that if $\lambda = \underline{\lambda}$ then g.l.b. $V[f(x)] < 0$. The remaining two lemmas show that the set of λ's is divided into two disjoint sets: those λ's for which the minimum of V is zero and those for which it is negative.

LEMMA 3. *If* $f(x) = 0$ *minimizes* V *for* $\lambda = \lambda_o$, *then only* $f(x) \equiv 0$ *minimizes* V *for all* $\lambda < \lambda_o$.

Proof:[†] We deduce directly from the form of V that if $\lambda < \lambda_o$, $V \geq 0$ for all A-functions. Suppose there is a $\lambda = \lambda^* < \lambda_o$ and an A-function $f^*(x) \not\equiv 0$ which minimizes V, i.e. $V[f^*(x);\lambda^*] = 0$. Then if λ is in the range $\lambda^* < \lambda < \lambda_o$, $V[f^*;\lambda] < V[f^*;\lambda^*] = 0$. This is in contradiction to the non-negative property of V for $\lambda < \lambda_o$ and the proof of the lemma is complete.

LEMMA 4. *If the minimum of* V *is negative for* $\lambda = \lambda_o$ *then it is negative for all* $\lambda \geq \lambda_o$.

Proof: By contradiction using Lemma 3.

The theorem is then proved if we define $\lambda_M(\rho)$ as the l.u.b. of those λ for which $f(x) = 0$ minimizes V. Thus for each ρ there is a $\lambda = \lambda_M$ such that for all $\lambda > \lambda_M$ there are buckled states with less energy than the unbuckled state and for $\lambda < \lambda_M$ all buckled states have greater energy than the unbuckled state.

The intermediate load can also be characterized by the

[†] This form of the proof was suggested by Dr. M. Newman.

minimum property,

$$(5.7) \qquad \lambda_M(\rho) = \min_{f(x) \neq 0 \in A} \left[\frac{\int_0^1 (2f'^2 + g'^2)x^3 dx}{\int_0^1 2f^2 x^3 dx} \right],$$

where $g'(x)$ is defined in (5.1). Upper bounds for λ_M are obtained by selecting trial A-functions to make the quotient in (5.7) as small as possible. For example, functions of the form $f(x) = \beta F(x)$ are considered, where $F(x) \in A$ is a specified function and β is an arbitrary constant. The quotient is then a function of β only and β is determined to minimize it. Some of the trial functions that were used and the resulting upper bounds, $\lambda_{M,i}(\rho) = a_i + b_i 10^{-3}\rho^2$, are shown* in Table I.

Table I

i	F(x)	$\lambda_{M,i} = a_i + b_i 10^{-3}\rho^2$	
		a_i	b_i
1	$1-x$	15.0	6.5166
2	$1-x^2$	16.0	3.7037
3	$1-x^3$	17.50	2.4451
4	$1-x^4$	19.20	1.7166
5	$1+x-2x^2$	16.7576	2.7836
6	$2(2-x-x^2)$	15.5054	4.6055

---

* The calculations were performed by M. Szeto.

The results* are summarized in Fig. 3 where we also show accurate approximations of λ_M obtained from a numerical solution of Problem B. Details of these and other calculations will be reported elsewhere [22].

6. Other Bifurcation Problems

Analogous results can be obtained for other bifurcation buckling problems. For example, for the boundary value problem consisting of (2.1), (2.3) and (2.5) which we call Problem B_1, we can show that

$$\omega_1^2 \leq \lambda_M(\rho) < \underline{\lambda}(\rho)$$

where $\underline{\lambda}$ is now the minimum eigenvalue of the linearized shell buckling theory using the boundary conditions (2.5) in place of (2.4). Upper bounds for $\lambda_M(\rho)$ are obtained from a formula similar to (5.7). These results are graphed in Fig. 3 with the predictions of λ_M obtained from a numerical solution of the boundary value problem [22].

---

* $\omega_1^2 \approx 14.684$

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Captions for Figures

Figure 1. Sketch of conjectured load deformation curve for bifurcation buckling. Here D is a representative deformation, e.g. the normal displacement of the cap's center.

Figure 2. Shell geometry.

Figure 3. Intermediate buckling loads for Problems B and B^*. The upper bound curves are essentially the envelopes of the curves given in Tables I and II.

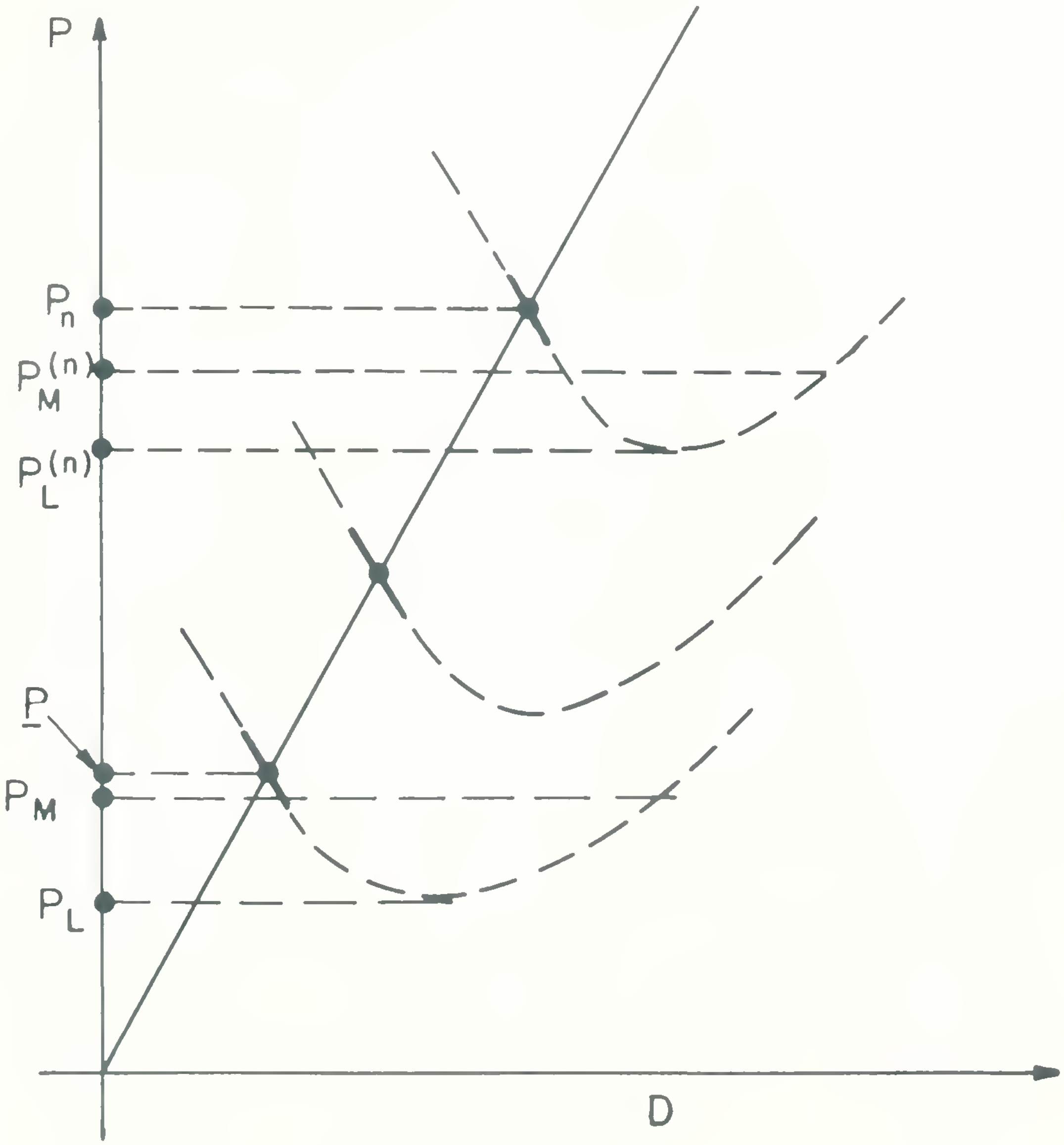

Figure I

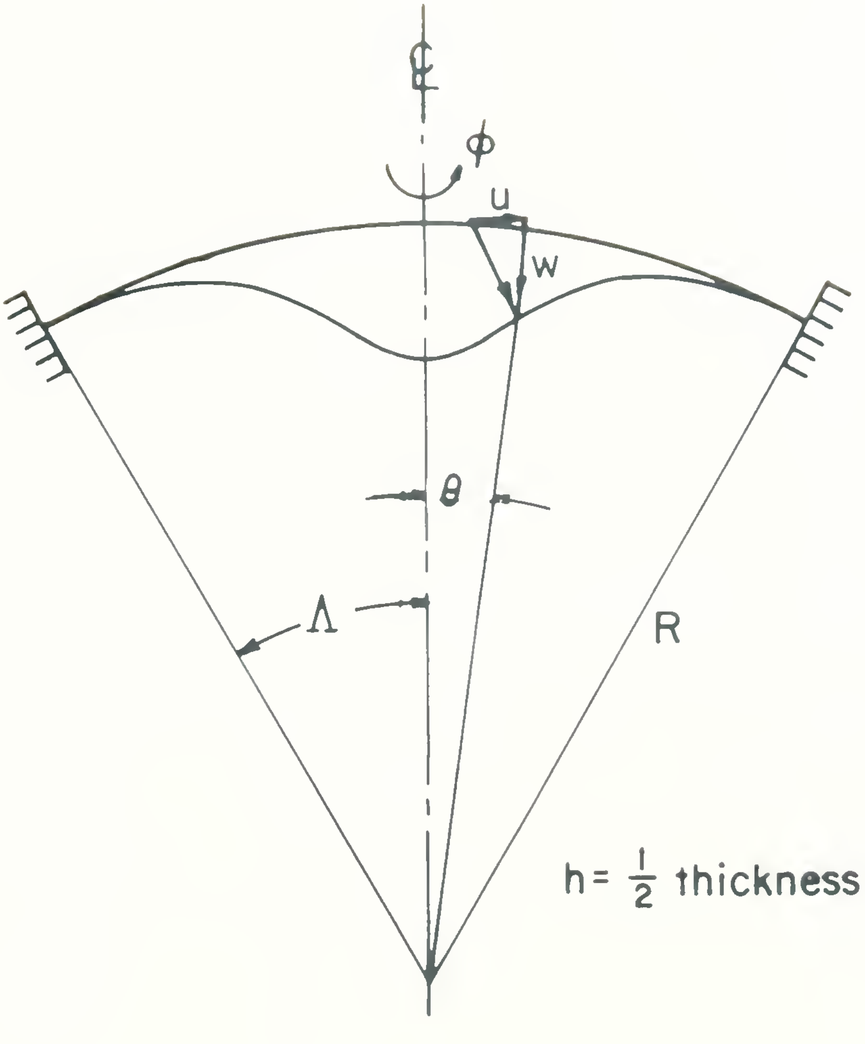

Figure 2

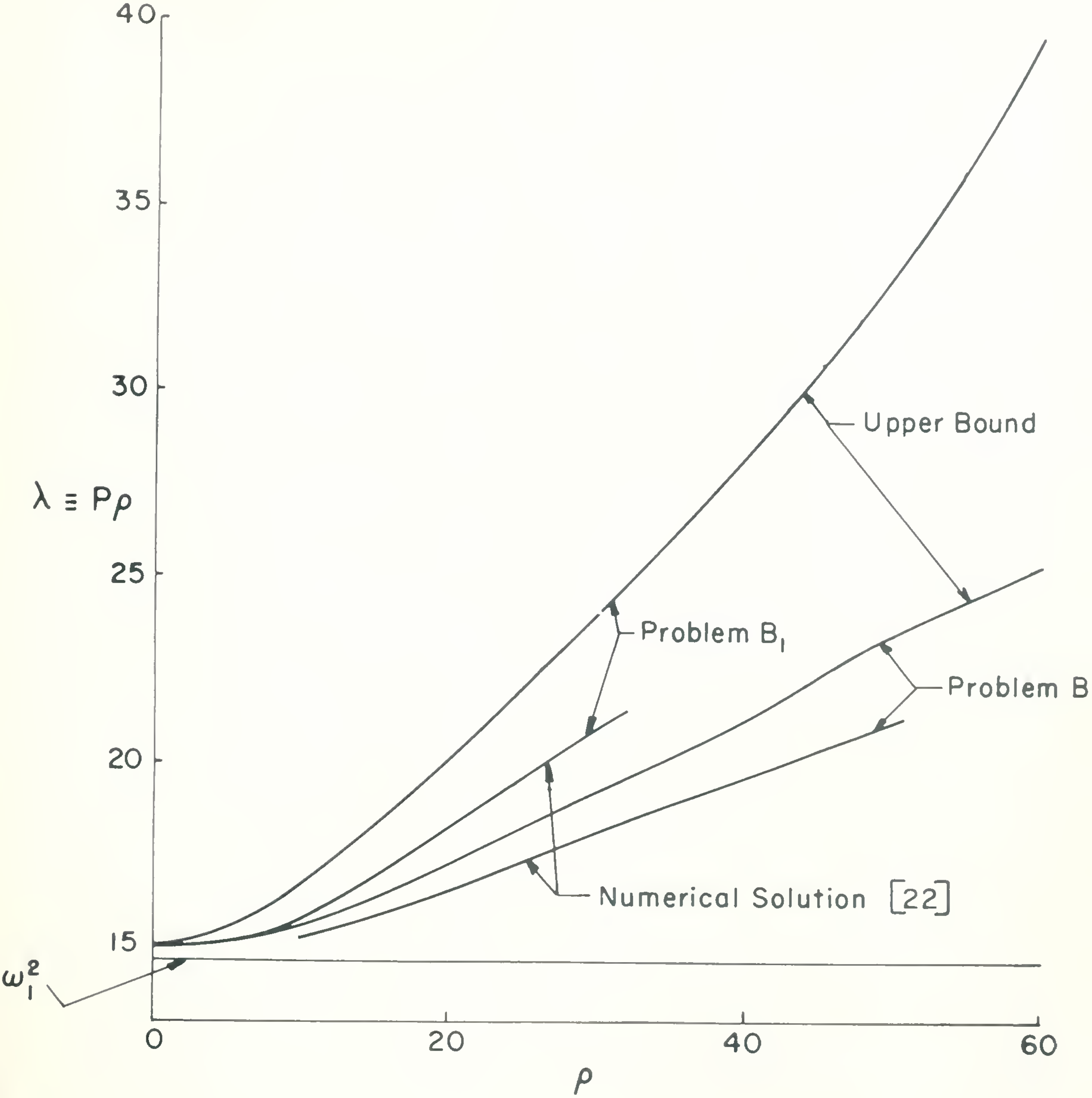
40
35
30
25
20
15
$\lambda \equiv P\rho$
ω_1^2
0
20
40
60
ρ
Upper Bound
Problem B_1
Problem B
Numerical Solution [22]

Figure 3

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